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# Structured note with American options to estimate the probability of default

## *Nota estructurada con opciones americanas para estimar la probabilidad de incumplimiento*

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**Abstract**

The objectives are: 1) the estimation of the default probability of six IPC issuers of the BMV and 2) the valuation of structured notes with American options applying discrete time stochastic dynamic programming, the results obtained are consistent with the Merton (1974) model, but American options allow early exercise. The scopes are: 1) the recursive binomial model is analyzed and 2) the probabilities of default are estimated for the first time with this model and the limitations are: 1) there are no implied volatility data, but historical volatility is used and 2) the loss given default and exposure at default are not estimated. The conclusions indicate that the proposed model allows early exercise to comply with the commitments and the valuation of the American options and the estimation of the default probabilities are consistent with the Merton structural model.

*JEL Code:* G13, D81, G32, C61
*Keywords:* option's pricing; risk management; stochastic dynamic programming

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**Resumen**

Los objetivos son: 1) estimar la probabilidad de incumplimiento de seis emisoras del IPC de la BMV y 2) la valuación de notas estructuradas con opciones americanas es aplicando la programación dinámica estocástica en tiempo discreto, los resultados son coherentes con el modelo de Merton (1974), pero las opciones americanas permiten el ejercicio anticipado. Los alcances son: 1) se presenta el modelo binomial recursivo y 2) las probabilidades de incumplimiento son estimadas por primera vez con este modelo y las limitaciones son: 1) no se tienen datos de la volatilidad implícita, pero es utilizada la volatilidad histórica y 2) no son estimadas: 1) la pérdida y la exposición por incumplimiento. Las conclusiones indican que el modelo propuesto permite el ejercicio anticipado para cumplir con los compromisos y las valuaciones de las opciones americanas y las estimaciones de las probabilidades de incumplimiento son coherentes con el modelo estructural de Merton.

*Código JEL:* G13, D81, G32, C61
*Palabras clave:* valuación de opciones; administración de riesgos; programación dinámica

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## Introduction

Credit risk has existed since loans were first negotiated and is currently related to the situation faced by: 1) financial institutions when they grant credit, and 2) economic agents when they make decisions to invest in bonds, stocks, promissory notes, or the sale of products or services on credit, because the agreements include commitments to pay and default by the counterparty is probable. The asset is estimated using the share price, and the probability of default is estimated using option valuation relative to the liability. The hypothesis is that if the exogenous factors (asset, liability, volatility, risk-free interest rate) and endogenous factors (term and settlement price) are known, then it is possible to estimate the probability of default with an American put option.

The risk-free interest rate is a theoretical interest rate for a risk-free investment. It is the interest rate used to discount expected, risk-free cash flows to the present. Substitutes for the risk-free interest rate are: 1) the Treasury rate, 2) the London Interbank Offered Rate (LIBOR), and 3) the overnight index swap (OIS) interest rate. An OIS is obtained through an interest rate swap (IRS), which exchanges the overnight interest rate for a fixed rate over a specified term.

The Basel Committee on Banking Supervision established the agreement in 1988 among G-10[1] banks to apply minimum capital standards for credit risk, which is defined as the risk that the counterparty will default on its obligations.

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[1] Belgium, Canada, France, Germany, Italy, Japan, Netherlands, Sweden, United Kingdom, and the United States

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Credit risk estimation models are based on economic and financial variables and use weighted factors to estimate default probabilities, following the work of Altman (1968) and Altman et al. (1977).

The Basel Committee on Banking Supervision presented recommendations in 2004 that are based on: 1) the calculation of minimum capital requirements, the credit quality of borrowers, and capital requirements for operational risk; 2) the supervision of capital management, and 3) market discipline for banking practices and standardization for the coverage of market, credit, operational, and legal risks. Furthermore, transparency standards were established, information on risk exposure was published, and statistics and credit risk estimation were defined with the probability of default, loss, and exposure to default.

Credit risk estimation is based on probability theory, statistics, and finance. Default risk estimation comprises four stages: first, a downgrade in credit rating; second, default; third, exposure; and fourth, recovery rate. Therefore, credit risk estimation considers three components: default probability, loss given default, and exposure at default. The difficulty in modeling credit risk arises because default is an infrequent event.

The structural models of Merton (1974), Black and Cox (1976), Geske (1977), Gupton et al. (1997), Jones et al. (1984), Vasicek (1984), Hull and White (1995), Longstaff and Schwartz (1995), Kealhofer (2003), Jarrow and Protter (2004), and Jorion (2007) for credit risk estimation consider default to be an endogenous variable that is influenced by capital structure and assume that: 1) investors have complete market information, 2) they have knowledge of the value of assets and liabilities, 3) changes in assets determine default, and 4) default occurs if assets are less than liabilities, i.e., if the company does not have sufficient assets to cover its liabilities. Therefore, structural models focus on modeling assets.

The reduced-form models of Litterman and Iben (1991), Jarrow and Turnbull (1995), Jarrow et al. (1997), Lando (1998), Madan and Unal (2000), Duffie (1998), Duffie and Singleton (1999), Hull and White (2001), and Jarrow and Protter (2004) for credit risk estimation consider default to be an exogenous variable and use the credit spread to estimate the probability of default with two types of models: 1) intensity and 2) credit migration, that is, they do not require information on the capital structure and assume that investors do not have: 1) complete information on the company and 2) knowledge of the asset or liability.

Credit risk models for estimating expected loss over a period and at a given confidence level were promoted by the Basel Committee on Banking Supervision in 2004, including the CreditMetrics models by Gupton et al. (1997) and CreditRisk+ by Credit Suisse First Boston (1997). Hybrid models for estimating credit risk are based on structural and reduced models.

The objective is to apply Climent Hernández's (2014) recursive binomial model for credit risk assessment, estimating the probability of default with the innovation of using an American put option with a national risk-free interest rate with collateral (TIIE, interbank equilibrium interest rate; Spanish: tasa de interés interbancaria de equilibrio) and the Greek kappa.

The paper is structured as follows: theoretical framework, methodology, results, discussion, conclusions, references, and finally an appendix.

## Theoretical framework

Credit risk is the probability that a debtor will default on their financial obligations, resulting in losses for the creditor. Understanding credit risk is important for the valuation of contingent liabilities, solvency analysis, the design of financial products with lower levels of risk, the structuring of derivative products, the evaluation of investment projects, the development of risk hedging strategies, the assessment of systemic risk, the design of public policies to mitigate credit risk, and the identification, measurement, monitoring, and control of credit risk.

In 2010, the Basel Committee on Banking Supervision presented regulations, prompted by the 2008 financial crisis, on bank capital adequacy and liquidity, which were agreed upon by the G-20.[2]

Hunzingera and Labuschagnec (2014) indicate that the binomial model is used for option valuation, and professionals and academics agree with the model using the discounting of expected cash flows with a risk-neutral measure and the risk-free interest rate.

Klieštika and Cúgb (2015) indicate that financial institutions are exposed to credit risk; therefore, it is necessary to improve models for identifying, estimating, and hedging credit risk. They compare the quantification of credit risk using the CreditRisk+, CreditMetrics, and Merton (1974) models.

Sierra Juárez (2017) proposes a hybrid model based on Merton's model (1974) and the Erdös and Rényi model (1960) to analyze contagion in the Mexican banking system, concluding that banks can be infected by an initial default and fall into default or not be affected by the event, depending on volatility and changes in liabilities.

Abid et al. (2020) explore the correlation between the probability of default, investment horizons, and ratings to enhance financial decision-making using the implied probability of default derived

[2] Argentina, Australia, Brazil, Canada, China, France, Germany, India, Indonesia, Italy, Japan, Mexico, Russia, Saudi Arabia, South Africa, South Korea, Spain, Turkey, United Kingdom, and the United States

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from credit default swap (CDS) spreads, and conclude that the CDS-based methodology is effective for assessing and predicting default risk.

Campolieti et al. (2022) analyze a structural model and a hybrid model, including the probability of default and the credit default swap spread, using a spectral expansion model, and conclude that the study offers innovations in credit risk modeling with practical applications.

Gubareva, M. (2021) provides a model based on market information to estimate expected loss on default in accordance with international financial reporting standards, applying actuarial models to estimate the probability of default, and concluding that it is possible to estimate expected loss.

Gredil et al. (2022) analyze the capacity of credit ratings and default risk measures using structural and reduced-form models, concluding that ratings complement risk measures by providing incremental information on default risk.

Hasnaoui and Hasnaoui (2022) assess the impact of human capital efficiency on commercial bank credit risk using default probability, the ratio of non-performing loans to total loans, and credit ratings, concluding that human capital efficiency has a strong negative impact on commercial bank credit risk.

Kevkhishvili (2022) presents an approach to detect and quantify changes in credit quality using a model based on long-term credit quality deterioration, applying a stochastic Uhlenbeck and Ornstein mean-reversion process to model leverage, demonstrating that the model is effective in detecting and quantifying changes in credit quality and probability of default, concluding that the endogenous change model is robust and efficient.

Sha (2022) analyzes how credit rating manipulation affects perceptions of loans, concluding that such manipulation is a considerable problem that poses challenges to accurate credit risk assessment.

Yamanaka and Kinoshita (2022) propose a structural credit risk model based on purchase order information and perform an empirical analysis using real data, using purchase order volume to estimate the asset value and the probability of default, concluding that the structural model is an effective tool for credit risk analysis.

Benhamed and Gassouma (2023) assess the ability of regulatory capital requirements to cover expected loss due to default, estimating the probability of default and simulating the sensitivity of systematic risk using the Monte Carlo method and loss distribution, concluding that regulatory capital is derived from economic capital, but with greater sensitivity to systematic risk, resulting in additional costs due to allocating more capital than necessary.

## Methodology

The scope of the work is to present a result for estimating the probability of default using the recursive binomial model for the valuation of American put options with the kappa Greek. The originality of the work lies in estimating the probability of default using an American put option.

The model proposed in this paper is a structural model that applies the valuation of American options proposed by Climent Hernández (2014) with the corresponding innovations. Default occurs when the asset $A_T$ is less than the liability $P_T$, that is, when $A_T < P_T$. Assuming that the liability $P_T$ is represented by a zero-coupon, risk-free bond $B_t$, if on the maturity date $A_T > B_T = P_T = S$, then the holder of the structured product $\Pi_T$ has the following portfolio: $\Pi_T = S - \max(S - A_T, 0) = S$.

If the company purchases the liability $P_0 = B_0 = S\exp(-iT)$ and acquires a call option on the asset $A_t$ with a settlement price S equal to the nominal value of the bond $B_T$, i.e., if on the maturity date $A_T < B_T = P_T = S$, then $\Pi_T = S + \max(A_T - S, 0) = S$, i.e., the company has the liability to meet its obligations. If $A_T > B_T = P_T = S$, then the company exercises the call option and recovers the asset, i.e., $\Pi_T = S + \max(A_T - S, 0) = A_T$.

The equivalent approach is: if the company purchases the liability $P_0 = B_0 = S\exp(-iT)$ and issues a put option on the asset $A_t$ with a settlement price S equal to the nominal value of the bond $B_T$, i.e., if $A_T > B_T = P_T = S$, then $\Pi_T = S - \max(S - A_T, 0) = S$, i.e., the company has the funds to meet its obligations. If $A_T < B_T = P_T = S$, then the company settles the put option at the bond's nominal value, $B_T = P_T = S$, and recovers the asset, i.e., $\Pi_T = S - \max(S - A_T, 0) = A_T$.

The structured product $\Pi_T$ is a portfolio consisting of: 1) the long position of a risk-free bond $B_0$ with nominal value $B_T$, where $B_0$ is equal to the liability $P_0$ invested at the risk-free interest rate i, i.e., $B_T = B_0\exp(iT)$, where the time elapsed is T, and 2) the short position of a put option on the asset $A_t$ with a settlement price S equal to $B_T$, i.e., $B_T = P_T = S$. Therefore, the value of the structured product with European put options $p(t, A_t)$ on the trading date is $\Pi_0 = P_0 - p(t, A_t)$ and on the expiration date is $\Pi_T = \min(A_T, S)$. The value of the structured product with American options on the trading date is $\Pi_0 = P_0 - P(t, A_t)$, and on the exercise or settlement date is $\Pi_{0 = P0 - P(t, At)}$

The equation that determines the value of the option (financial insurance) is critical in structural models. The derivation of the equation considers the following assumption:

$$\Delta A_t = \mu_A A_t \Delta t + \sigma_A A_t \Delta W_{At} \tag{1}$$

where $\Delta A_t$ is the change in the asset, $\mu_A$ is the expected interest rate, $\sigma_A$ is the standard deviation of the asset, and $\Delta W_{At}$ is the change in the Wiener process.

The assumption that the change in the asset is described by the stochastic differential equation (SDE) represented by equation (1) means that the asset is an exogenous variable in an efficient market with the following hypotheses: 1) successive changes are stochastically independent, and 2) successive changes are governed by a probability distribution; therefore, the changes are discrete and independent. The asset has two components: 1) deterministic and 2) stochastic. The deterministic component corresponds to the asset change model, and the stochastic component is represented by the Wiener process. The variables $\mu_A$, $\sigma_A$, and $A_t$ are not easy to observe because financial statements are published monthly or quarterly.

If the asset $A_t$ is a variable that is difficult to observe, then assuming that there is financial insurance $V_t$, where the theoretical value of the financial insurance is a function of time and of the asset, that is, $V_t = f(t, A_t)$, therefore, the change in the valuation of the financial insurance is modeled with the following EDE:

$$\Delta V_t = \mu_V V_t \Delta t + \sigma_V V_t \Delta W_{Vt} \tag{2}$$

where $\Delta V_t$ is the change in the valuation of the financial insurance, $\mu_V$ is the expected interest rate, $\sigma_V$ is the standard deviation of the financial insurance, and $\Delta W_{Vt}$ is the change in the Wiener process.

The analogy between the change in asset $\Delta A_t$ and the change in financial insurance valuation $\Delta V_t$ exists. If $\Delta V_t$ is determined simultaneously by equation (2) and the functional ratio $f(t, A_t)$, then:

$$\Delta V_t = \left( \frac{\Delta f(t, A_t)}{\Delta t} + \frac{\mu_A A_t \Delta f(t, A_t)}{\Delta A_t} + \frac{(\sigma_A A_t)^2 \Delta^2 f(t, A_t)}{2\Delta A_t^2} \right) \Delta t + \frac{\sigma_A A_t \Delta f(t, A_t) \Delta W_{At}}{\Delta A_t} \tag{3}$$

If $\Delta W_t = \Delta W_{Vt} = \Delta W_{At}$ and there is also a strong positive correlation between equations (1) and (2), i.e., if $\rho(V_t, A_t) \rightarrow 1$, then the change in the value of the portfolio is:

$$\Delta X_t = (\mu_A \omega_1 + \mu_V \omega_2) \Delta t + (\sigma_A \omega_1 + \sigma_V \omega_2) \Delta W_t \tag{4}$$

where $w_k$ is the amount invested in the components of portfolio $X_t$. The valuation of financial insurance $V_t$ applies the theory of arbitrage with a portfolio that guarantees that the coefficient of the stochastic member is zero, so the return on the portfolio is deterministic because the impact of the Wiener process is eliminated, the portfolio is risk-free, and to avoid arbitrage, the expected return on the portfolio is zero. Therefore, the conditions are:

$$\begin{aligned} \sigma_A \omega_1 + \sigma_V \omega_2 &= 0 \\ \mu_A \omega_1 + \mu_V \omega_2 &= 0 \end{aligned} \tag{5}$$

where equation (5) is a risk-free portfolio, a no-arbitrage portfolio, and is a system of two equations with two variables, if it satisfies the following condition:

$$\frac{\mu_A}{\sigma_A} = \frac{\mu_V}{\sigma_V} \tag{6}$$

then:

$$if(t, A_t) = \frac{\Delta f(t, A_t)}{\Delta t} + \frac{\mu_A A_t \Delta f(t, A_t)}{\Delta A_t} + \frac{(\sigma_A A_t)^2 \Delta^2 f(t, A_t)}{2\Delta A_t^2} \tag{7}$$

where I is the risk-free interest rate, t is time, f (t, $A_t$) is the value of the financial insurance issued on asset $A_t$, $\mu_A$ is the expected interest rate, and $\sigma_A$ is the volatility of the asset. Equation (7) is the value of the contingent payment $V_t$, so the contingent payment $V_t$ does not depend on $\mu_V$. The distribution is considered risk-neutral for both the valuation of $V_t$ and the value of $A_t$. The valuation procedure is a function of time and of the asset.

The assumption is that asset $A_t$ is composed of the following two types of liabilities:

1. Liability $P_t$ of the company expressed by a zero-coupon bond $B_t$ at time t
2. The capital of the company $C_t$, expressed as the product of the share price $M_t$ and the number of shares issued N, i.e., $C_t$ = N$M_t$. Therefore, the company's balance sheet is represented as follows:

$$A_t = P_t + C_t \tag{8}$$

where $A_t$ is the asset, $P_t$ is the liability, and $C_t$ is capital.

The issuance of a $B_t$ bond is a commitment made by the company to pay the nominal value of the $B_T$ bond to the holder at time T, so $B_T$ = S, where S is the option's settlement price. If the company does not issue any more bonds and if, in addition, the company does not have the funds to pay its creditors at time T, then this is equivalent to the asset $A_t$ being less than the nominal value of the $B_T$ bond = S, i.e., default occurs when $A_T < S$.

If the company has resources, then it pays creditors the nominal value of the zero-coupon bond, i.e., $B_T$, and if it does not have resources, then it defaults on its obligations to pay the nominal value of the zero-coupon bond, i.e., the company acts in the interests of shareholders. Therefore, the final condition is:

$$B_T = \min\left(A_T, S\right) \tag{9}$$

The conditions of equations (8) and (9) indicate that capital $C_t$ is an option on asset $A_t$ with settlement price S and satisfies:

$$iC_t = \frac{\Delta C_t}{\Delta t} + \frac{\mu_A A_t \Delta C_t}{\Delta A_t} + \frac{\left(\sigma_A A_t\right)^2 \Delta^2 C_t}{2\Delta A_t^2} \tag{10}$$

where i is the risk-free interest rate, $C_t$ is the capital, t is time, $\mu_A$ is the expected interest rate, and $\sigma_A$ is the volatility of the asset. The final condition is:

$$\begin{aligned} C_T &= \text{máx}\left(A_T - S, 0\right) \\ -C_T &= \max\left(S - A_T, 0\right) \end{aligned} \tag{11}$$

The approach is similar for the valuation of a call or put option on asset $A_t$ in discrete time because the random path of an independent binomial process converges to the Wiener process.

## *Capital as an American call option*

If $A_t = P_t + C_t$, then $C_t = A_t - P_t$, so capital $C_t$ is equivalent to an American call option $C(t, A_t)$, where the price of European call options $c(t, A_t)$ and the price of American call options $C(t, A_t)$ are theoretically equal, i.e., $c(t, A_t) = C(t, A_t)$.

## *Liability as a European put option*

If $P_t = A_t - C_t$, then the structured note with American options is $\Pi_t = A_t - C(t, A_t)$, whereas the structured note with European options is $\Pi_t = A_t - c(t, A_t)$:

$$\Pi_t = A_t\tilde{B}\left(n-w,\tilde{\theta}\right) + P_tB\left(n-w,\theta\right) \rightarrow A_t\Phi\left(-d_1\right) + P_t\Phi\left(d_2\right) \tag{12}$$

where the complementary binomial model converges to Merton's (1974) model.

Similarly, the structured note with American options is $\Pi_t = P_t - P(t, A_t)$, where the theoretical price of European put options $p(t, A_t)$ is lower than the theoretical price of American put options $P(t, A_t)$, i.e., $p(t, A_t) < P(t, A_t)$, then the structured note $P_t - P(t, A_t) < P_t - p(t, A_t)$ and due to the parity of purchase and sale, $A_t - c(t, A_t) = P_t - p(t, A_t)$ is satisfied. Therefore, it is necessary to apply the recursive model for the valuation of American put options.

## *Recursive binomial model for the valuation of American options*

The recursive binomial model enables the valuation of American put options for a structured product with American options that grants the possibility of exercising the American option early to meet obligations. The recursive model for valuing American put options is based on the model developed by Climent Hernández (2014):

$$V_\eta^{a^k d^{\eta-k}} = \begin{cases} \max\left(\left(V_\eta^{a^{k+1}d^{\eta-k}}\pi + V_\eta^{a^k d^{\eta-k+1}}\theta\right)\exp\left(-i\delta T\right), V_\eta^{a^k d^{\eta-k}}\right) & \text{si } 0 \le k \le \eta < n \\ V_\eta^{a^k d^{\eta-k}} = \max\left(S - M_0 a^k d^{\eta-k}, 0\right) & \text{si } 0 \le k \le \eta = n \end{cases} \tag{13}$$

where $\delta = n^{-1}$, $u = \exp(idT)$, $0 < d = a^{-1} < 1 < a = -2^{-1}B + 2^{-1}(B^2 - 4)^{0.5}$, for the value of the coefficient $B = -(\sigma^2\Delta t + u^2 + 1)u^{-1}$, $\pi = (u - d)(a - d)^{-1}$ and $\theta = 1 - \pi$. The estimation of the parameters u, a, d, π, and θ is presented in the Appendix: equations (26), (30), and (27), respectively.

Therefore, the valuation of the structured note with American options is:

$$\Pi_t = P_t - V_\eta^{a^k d^{\eta-k}} = P_t - P\left(t, A_t\right) \tag{14}$$

where $\Pi_t$ is the structured note with American options, $P_t$ is the liability, and $P(t, A_t)$ is the American put option issued on the asset $A_t$ with settlement price S.

### *Greek letter concept for the recursive binomial model*

Delta is the proportional change in the option's valuation relative to the proportional change in the underlying price and represents short-run hedging:

$$\Delta = \frac{\Delta V\left(t, A_t\right)}{\Delta A_t} = \frac{V\left(t, A_t^a\right) - V\left(t, A_t^d\right)}{A_t^a - A_t^d} \tag{15}$$

The range is the proportional change in delta relative to the proportional change in the underlying price and represents long-run coverage:

$$\Gamma = \frac{\Delta^2 V\left(t, A_t\right)}{\Delta A_t^2} = \frac{2\left(V\left(t, A_t^{a^2}\right) - V\left(t, A_t^{d^2}\right)\right)}{A_t^{a^2} - A_t^{d^2}} \tag{16}$$

Nu is the proportional change in the option's valuation relative to the proportional change in the underlying volatility:

$$\nu = \frac{\Delta V\left(t, A_t\right)}{\Delta \sigma} = \frac{V\left(t, \sigma_2\right) - V\left(t, \sigma_1\right)}{\sigma_2 - \sigma_1} \tag{17}$$

Rho is the proportional change in the option's valuation relative to the proportional change in the risk-free interest rate:

$$\rho = \frac{\Delta V\left(t, A_t\right)}{\Delta i} = \frac{V\left(t, i_2\right) - V\left(t, i_1\right)}{i_2 - i_1} \tag{18}$$

Theta is the proportional change in the option's valuation relative to the proportional change in time elapsed:

$$\Theta = \frac{\Delta V\left(t, A_t\right)}{\Delta t} = \frac{V\left(t, A_t^{ad}\right) - V\left(t, A_t\right)}{2\delta T} \tag{19}$$

Kappa is the proportional change in the option's valuation relative to the proportional change in the settlement price:

$$\kappa = \frac{\Delta V(t, A_t)}{\Delta S} = \frac{V(t, S_2) - V(t, S_1)}{S_2 - S_1} \tag{20}$$

## *Probability of default*

The probability of default is equivalent to the European call option being out of the money or the European put option being in the money:

$$P\ (A_T < S) = \kappa \exp(iT) \tag{21}$$

## Results

The Price and Quotations Index (IPC; Spanish: Índice de Precios y Cotizaciones) was launched on October 30, 1978, and measures the performance of the most liquid stocks listed on the Mexican Stock Exchange (BMV; Spanish: Bolsa Mexicana de Valores) to provide a representative and replicable index of the Mexican stock market. The relative frequencies fr and cumulative relative frequencies Fr of the six issuers with the highest weightings as of June 28, 2023, are presented in Table 1.

Table 1
Frequencies of IPC weights

| Issuer | Name | *fr* | *Fr* |
|---|---|---|---|
| WALMEX | Wal Mart de México SAB de CV | 0.1974 | 0.1974 |
| AMXB | América Móvil SAB de CV | 0.1267 | 0.3241 |
| GMEXICOB | Grupo México SAB de CV | 0.1004 | 0.4245 |
| GFNORTEO | Grupo Financiero Banorte SAB de CV | 0.0637 | 0.4882 |
| BIMBOA | Grupo Bimbo SAB de CV | 0.0602 | 0.5484 |
| FEMSAUBD | Fomento Económico Mexicano SAB de CV | 0.0560 | 0.6044 |

Source: created by the authors using information from Refinitiv, 2023

Table 1 shows the relative frequencies (weighting with respect to the total) of the IPC in descending order until they accumulate more than 60%: WALMEX 19.74 %, AMXB 12.67 %, GMEXICOB

10.04 %, GFNORTEO 6.37 %, BIMBOA 6.02 % and FEMSAUBD 5.60 %. In other words, these are the six most liquid issuers of the 35 that make up the IPC, where the weightings are a function of market capitalization value and represent more than 5%. The Pareto chart of the weightings is presented in Figure 1.

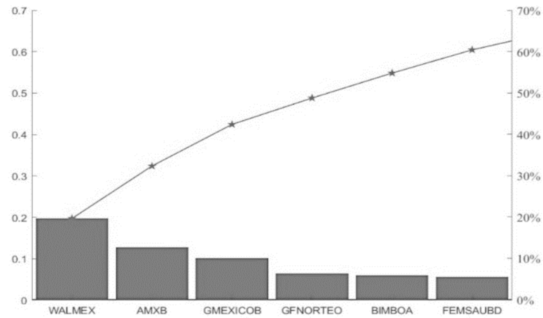


Figure 1. Pareto chart of IPC weights
Source: created by the authors using information from Refinitiv, 2023

Figure 1 shows the Pareto chart to justify that WALMEX, AMXB, GMEXICOB, GFNORTEO, BIMBOA, and FEMSAUBDare the six issuers with a significant weighting (greater than 5%) and represent more than 60% of the IPC, i.e., they are the six issuers with the highest weightings in relation to the totalIPC.

Therefore, the selection of the six issuers of the IPC from the BMV is based on the criteria of relevance and representativeness because the six issuers with a significant weighting (greater than 5%) in the IPC were chosen. The 4 136 prices of the six issuers from January 2, 2007, to June 28, 2023, are presented in Figure 2.


Figure 2. Daily prices
Source: created by the authors using information from Yahoo Finance, 2023

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Figure 2 shows the prices of issuers during the study period; the returns are positive, i.e., prices on January 2, 2007, are lower than those on June 28, 2023. The 4 135 returns from January 2, 2007, to June 28, 2023, are presented in Figure 3.

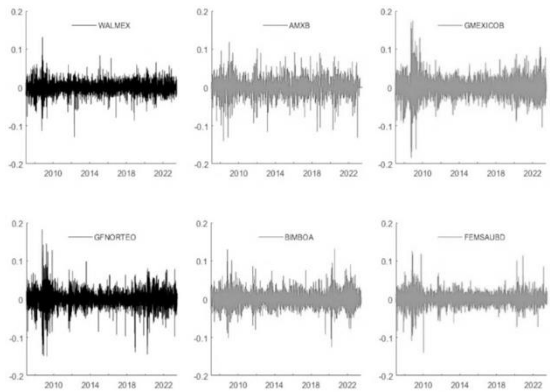


Figure 3. Daily logarithmic returns
Source: created by the authors using information from Yahoo Finance, 2023

Figure 3 shows the 4 135 daily logarithmic returns of the six issuers in the period from January 3, 2007, to June 28, 2023. The 4 134 historical volatilities from January 4, 2007, to June 28, 2023, are presented in Figure 4.

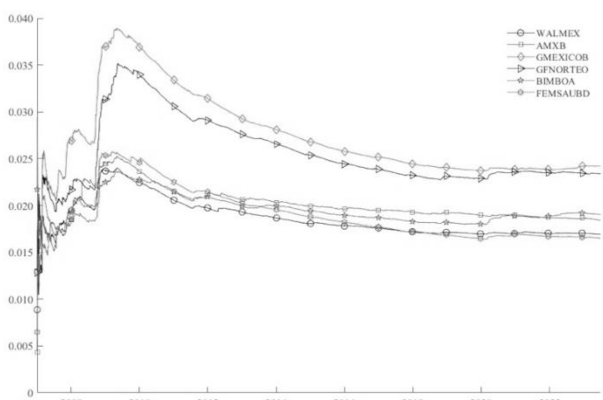


Figure 4. Daily historical volatilities
Source: created by the authors using information from Yahoo Finance, 2023

Figure 4 shows that historical volatilities increased during 2008 and 2009, then decreased until 2011, when a further increase was observed, followed by continued decline until 2020, when another increase was observed. The issuer with the highest volatility after 2022 is GMEXICOB, followed by GFNORTEO, BIMBOA, AMXB, WALMEX, and FEMSAUBD. The descriptive statistics for the 4 135 returns are presented in Table 2.

Table 2
Descriptive statistics

| Issuer | Mean | Median | min | max | $s_x$ | $g_1$ | $g_2$ |
|---|---|---|---|---|---|---|---|
| WALMEX | 0.000256 | 0.000279 | -0.1298 | 0.1319 | 0.0169 | -0.19 | 4.13 |
| AMXB | 0.000113 | 0.000000 | -0.1403 | 0.1186 | 0.0184 | -0.44 | 7.82 |
| GMEXICOB | 0.000461 | 0.000455 | -0.1841 | 0.1742 | 0.0242 | -0.10 | 5.59 |
| GFNORTEO | 0.000287 | 0.000530 | -0.1496 | 0.1823 | 0.0234 | -0.26 | 5.96 |
| BIMBOA | 0.000464 | 0.000000 | -0.1255 | 0.1317 | 0.0190 | 0.25 | 3.75 |
| FEMSAUBD | 0.000361 | 0.000000 | -0.1403 | 0.1258 | 0.0165 | 0.01 | 7.35 |

Source: created by the authors using information from Yahoo Finance, 2023

Table 2 presents the point estimates of the descriptive statistics (mean, median, minimum, maximum, standard deviation, skewness coefficient, and kurtosis) of the returns. The arithmetic means, skewness coefficients, and kurtosis coefficients confirm a positive trend, negative skewness for WALMEX, AMXB, GMEXICOB, and GFNORTEO, positive skewness for BIMBOA and FEMSAUBD, and leptokurtosis in the distributions of returns. The box plots of returns are presented in Figure 5.

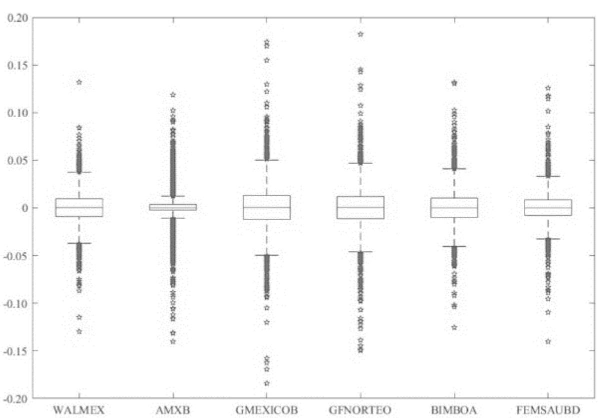


Figure 5. Box and whisker plots
Source: created by the authors using information from Yahoo Finance, 2023

Figure 5 presents the box plots of returns, confirming that the medians are positive and the skewness is negative for WALMEX, AMXB, GMEXICOB, and GFNORTEO, the skewness is positive for BIMBOA and FEMSAUBD, and the leptokurtosis of the return distributions.

If the TIIE funding rate on March 31, 2023, for 90 days is $i^{(4)} = 0.115221$ based on 360 days, then the instantaneous annual risk-free interest rate based on 365 days is:

$$i_{\infty} = \ln\left(1 + \frac{365 i^{(m)}}{90m}\right) = \ln\left(1 + \frac{365 i_m}{90}\right) \tag{22}$$

Therefore, the instantaneous annual risk-free interest rate is i = 0.110486517732013. If the option's maturity date is 06/29/2023, then the time to maturity is 90 days, i.e., T = 0.246575. The asset, liability[3], and settlement price of the six issuers as of 03/31/2023 are presented in Table 3.

Table 3
Assets, liabilities, and settlement price

| Issuer | $A_t$ | $P_t$ | S |
|---|---|---|---|
| WALMEX | 413 519 190.00 | 249 047 541.00 | 255 925 671.57 |
| AMXB | 1 593 341.00 | 1 184 070.00 | 1 216 771.34 |
| GMEXICOB | 20 174 276.00 | 9 545 370.00 | 9 808 991.56 |
| GFNORTEO | 2 130 031.00 | 1 877 460.00 | 1 929 311.20 |
| BIMBOA | 338 205.00 | 224 946.00 | 231 158.50 |
| FEMSAUBD | 810 692.00 | 449 552.00 | 461 967.61 |

Source: created by the authors, 2023

Table 3 shows the asset, liability, and settlement price, calculated using the liability, the risk-free interest rate, and the term, i.e., $S = P_t \exp(iT)$. The annual historical volatilities as of March 31, 2023, are shown in Table 4.

Table 4
Annual historical volatilities

| Factor | WALMEX | AMXB | GMEXICOB | GFNORTEO | BIMBOA | FEMSAUBD |
|---|---|---|---|---|---|---|
| $s_1$ | 0.269653 | 0.294685 | 0.384389 | 0.372283 | 0.303251 | 0.263513 |

Source: created by the authors using information from Yahoo Finance, 2023

[3]Assets and liabilities were obtained, respectively, for the first quarter of 2023 from the following pages: https://www.walmex.mx, https://www.americamovil.com, https://www.gmexico.com, https://www.grupobimbo.com, https://www.banorte.com and https://www.femsa.com.

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Table 4 shows the historical annual volatilities as of March 31, 2023, estimated from the standard deviation $s_x$ of returns, i.e., $s_1 = s_x 252^{0.5}$. The valuation of American put options using the proposed model (n = 5 000) and the valuation of European put options are presented in Table 5.

Table 5
Valuation of options

| Option | WALMEX | AMXB | GMEXICOB | GFNORTEO | BIMBOA | FEMSAUBD |
|---|---|---|---|---|---|---|
| $P(t, A_t)$ | 782.31 | 1 615.20 | 26.89 | 56 021.05 | 43.57 | 0.05 |
| $p(t, A_t)$ | 771.95 | 1 575.90 | 26.71 | 54 248.86 | 42.78 | 0.05 |

Source: created by the authors, 2023

Table 5 presents the valuation of American put options $P(t, A_t)$ using the recursive model proposed by Climent Hernández (2014) and the theoretical valuation of European put options $p(t, A_t)$ using the Merton model (1974). The theoretical prices of American options are higher (WALMEX 1.34 %, AMXB 2.49 %, GMEXICOB 0.66 %, GFNORTEO 3.27 %, BIMBOA 1.85 % and FEMSAUBD 0.66 %) than the theoretical prices of European options.

The theoretical price of American options is higher than the theoretical price of European options, but American options have the advantage of allowing early exercise of the structured note, i.e., if the company is unable to meet its commitments, i.e., if $P_t > A_t$, then it settles the liability and the American option early in order to meet its commitments. Therefore, the structured product with American options is more economical than the European structured product because American options have a higher price than European options. Both structured notes, American and European, provide the hedge to minimize default, but the structured note with American options allows early exercise when default is detected, without the need to wait until the maturity date, as in the case of the structured note with European options. The probability of default for the proposed model and the Merton (1974) model is presented in Table 6.

Table 6
Probability of default

| Factor | WALMEX | AMXB | GMEXICOB | GFNORTEO | BIMBOA | FEMSAUBD |
|---|---|---|---|---|---|---|
| $\Pi (A_T < S)$ | 0.0001 | 0.0255 | 0.0001 | 0.2924 | 0.0043 | 0.0000 |
| $\Phi (-d_2)$ | 0.0001 | 0.0253 | 0.0001 | 0.2775 | 0.0042 | 0.0000 |

Source: created by the authors, 2023

Table 6 presents the probability of default $\Pi (A_T < S)$ of the proposed model, which is similar to the probability of default $\Phi (-d_2)$ of the Merton model. Therefore, the proposed model evaluates the probability of default in a manner similar to the Merton model. The theoretical value of structured notes with American and European options is presented in Table 7.

Table 7
Value of structured notes

| Note | $\Pi_t = P_t - P(t, A_t)$ | $\Pi_t = P_t - p(t, A_t)$ |
|---|---|---|
| WALMEX | 249 046 758.69 | 249 046 769.05 |
| AMXB | 1 182 454.80 | 1 182 494.10 |
| GMEXICOB | 9 545 343.11 | 9 545 343.29 |
| GFNORTEO | 1 821 438.95 | 1 823 211.14 |
| BIMBOA | 224 902.43 | 224 903.22 |
| FEMSAUBD | 449 551.95 | 449 551.95 |

Source: created by the authors

Table 7 shows that the theoretical value of structured notes with American options is lower than the value of European structured notes because the theoretical price of American options is higher than the theoretical price of European options.

## Discussion

The valuation of derivative products is critical for informed decision-making. This analysis presents the application of a novel model for valuing structured notes using American put options, focusing on estimating the default probabilities of six issuers of the IPC listed on the BMV. The application of discrete-time stochastic dynamic programming and the recursive binomial model represent a relevant contribution to the theory and practice of credit risk management.

The results obtained from applying the recursive binomial model and discrete-time stochastic dynamic programming are consistent with existing theory on American options and Merton's (1974) structural model for credit risk. The derived analytical equation provides an effective tool for the early exercise of structured notes. Therefore, the proposed model achieves remarkable accuracy in estimating default probabilities, aligns with the theoretical expectations of Merton's (1974) model, considers the underlying economic fundamentals, and provides a detailed view of credit risk.

The study indicates that, despite the lack of implied volatility, the proposed model is robust for early valuation of financial commitments and provides reliable estimates of default probability. The model assumes a constant market structure and economic conditions, which does not reflect reality in volatile environments, but the recursive binomial model can incorporate deterministic or stochastic rates and volatilities.

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Future research could explore the integration of implied volatility data (volatility smile), real-time economic impacts to further refine credit risk models, or the application of other distribution functions.

## Conclusions

This paper presents an innovative model for valuing structured notes with American options. It applies discrete-time stochastic dynamic programming, a robust tool for valuing American put options, thereby capturing the complexity of these financial products in scenarios where early exercise of the American options in the portfolio is possible.

The analysis of the recursive binomial model indicates that it satisfies the conditions for: 1) the valuation of American options, 2) the estimation of default probabilities. Therefore, with a structured note and American options, a structural credit risk model is proposed, thereby achieving the objectives. The recursive binomial model is based on a Wiener process that satisfies the Black-Scholes partial differential equation in discrete time, and is adapted to allow early exercise of American put options. This approach is particularly relevant for credit risk management because it enables the estimation of default probability, providing information for decision-making on capital allocation and the implementation of hedging strategies.

If exogenous and endogenous factors are well known, it is possible to estimate the probability of default using an American put option and to use implied or historical volatilities in the proposed model. Therefore, the hypothesis is achieved with the proposed methodology.

The originality of the study lies in estimating the probability of default using the valuation of American put options, applying the concept of Greek letters (kappa) that justifies the application of the model, and estimating the probability of default with the advantage that American options can be exercised early, which gives the proposed model an additional quality compared to models with European options.

The proposed methodology is based on simplified assumptions and does not fully capture the complexity of credit risk. Thus, it is possible to underestimate or overestimate the probability of default and make inappropriate decisions in risk management because the proposed model is based on the valuation of options with historical data without comprehensively addressing aspects of credit risk, such as the correlation between returns, portfolio diversification, and macroeconomic indicators, to improve the accuracy of default probabilities.

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## Annex

### *Binomial model*

Climent Hernández (2014) proposed a model in which performance is modeled as a random walk and option valuation is performed using discrete-time stochastic dynamic programming to maximize the present value of probable cash flows and to economically quantify the risk the issuer considers when granting coverage through the option contract. The assumption is that at the end of each state (node) into which the term T is divided, there are two probable values based on the underlying price $M_0$ and, consequently, two values of the European option v(0, $M_0$), where the option's term is divided into n homogeneous intervals. Then, on the maturity date T, there are n + 1 probable values of the underlying price and n + 1 probable values of the option. In each state, the price increases with probability π from $M_t$ to $M_t a$ or decreases with probability θ = 1 – π from $M_t$ to $M_t d$, where 0 < d < 1 < a.

If the probable underlying prices and option values on the maturity date are well known, then the present value of the option is estimated using a replicating portfolio (synthetic European option) composed of: 1) the long position of Δ underlying assets at present value at the dividend interest rate or the foreign risk-free interest rate, that is:

$$\Delta M_t = \begin{cases} \Delta M_{t-1} a \exp(-rT) & \text{si } M_t \text{ aumenta} \\ \Delta M_{t-1} d \exp(-rT) & \text{si } M_t \text{ disminuye} \end{cases} \tag{23}$$

and 2) the short position of the option that needs to be valued, that is:

$$-v(0, M_0) = \begin{cases} -\max(M_T - S, 0) & \text{si es una opción de compra} \\ -\max(S - M_T, 0) & \text{si es una opción de venta} \end{cases} \tag{24}$$

then the hedge Δ that keeps the portfolio $\Pi_t = \Delta M_t - v(0, M_0)$ risk-free at time t is:

$$\Delta_t = \frac{v(t, M_t^a) - v(t, M_t^d)}{M_t^a - M_t^d}$$

(25)

where $\Delta_t$ is the optimal strategy, v(t, $M_t^a$) is the value of the European option when the price increases, v(t, $M_t^d$) is the value of the European option when the price decreases, $M_t^a$ is the underlying price when it increases, and $M_t^d$ is the underlying price when it decreases at time t.

If π is the probability that the underlying price will increase in each state, then the expected underlying return at the end of the interval δT is:

$$\mathrm{E}(X) = a\pi + d(1-\pi) = \exp(\mu\delta T) = u$$

(26)

where E(X) is the expected return in the interval δT, where $\delta = n^{-1}$. The risk-neutral probabilities (martingales) are:

$$\pi = \frac{u-d}{a-d} \quad y \quad 1-\pi = \frac{a-u}{a-d}$$

(27)

where $\delta = n^{-1}$ and $0 < d < 1 < a$. The proportional variance in the interval δT is:

$$\sigma^2 \delta T = \mathrm{Var}(X) = (a-d)^2 \pi(1-\pi) = (u-d)(a-u)$$

(28)

where $\delta = n^{-1}$ y $0 < d < 1 < a$.

The system of equations (26) and (28) has an infinite number of solutions. If ad = 1, then solving for the variable a yields the following equation:

$$a^2 - (\sigma^2 \delta T + u^2 + 1)u^{-1}a + 1 = 0$$

(29)

where A = 1, $B = -(\sigma^2\delta T + u^2 + 1)u^{-1}$, and C = 1, then the solution is:

$$a = \frac{-B + \sqrt{B^2 - 4}}{2} \quad y \quad d = \frac{-B - \sqrt{B^2 - 4}}{2}$$

(30)

Climent Hernández (2014) applied this solution to the valuation of American put options and compared the results with other models.

## *General model for the valuation of European options*

The general model for valuing European options is:

$$v(0,M_0)=\left(\sum_{k=0}^{n}\binom{n}{k}\pi^k(1-\pi)^{n-k}\,v\left(n,M_n^{a^k d^{n-k}}\right)\right)\exp(-iT)$$
$$v\left(n,M_n^{a^k d^{n-k}}\right)=\begin{cases}\max\left(Ma^k d^{n-k}-S,0\right) & \text{si es una opción de compra}\\ \max\left(S-Ma^k d^{n-k},0\right) & \text{si es una opción de venta}\end{cases} \tag{31}$$

where $\delta = n^{-1}$ y $0 < d < 1 < a$.

## *Complementary binomial model*

European options are in the money if the minimum integers w satisfy the following conditions:

$$\begin{aligned} Ma^w d^{n-w} &> S \quad \text{si es una opción de compra}\\ Ma^w d^{n-w} &< S \quad \text{si es una opción de venta}\end{aligned} \tag{32}$$

where the integers w of increments are:

$$\begin{aligned}\min(w) &> \left\lceil \frac{n\log(a)+\log(S)-\log(M)}{2\log(a)}\right\rceil \quad \text{si es una opción de compra}\\ \max(w) &< \left\lfloor \frac{n\log(d)-\log(S)+\log(M)}{2\log(d)}\right\rfloor \quad \text{si es una opción de venta}\end{aligned} \tag{33}$$

then the distribution function of the complementary binomial model is:

$$B(w,n,\pi)=\sum_{k=w}^{n}\binom{n}{k}\pi^k\theta^{n-k}=\sum_{k=0}^{n-w}\binom{n}{k}\theta^k\pi^{n-k}=B(n-w,\theta) \tag{34}$$

If $\tilde{\pi}=a\pi\exp(-i\delta T)$ y $\tilde{\theta}=1-\tilde{\pi}=d(1-\pi)\exp(-i\delta T)$, then:

$$\tilde{\pi}=\frac{a-u^{-1}}{a-d} \quad y \quad \tilde{\theta}=1-\tilde{\pi}=\frac{u^{-1}-d}{a-d} \tag{35}$$

Therefore, the complementary binomial model for the valuation of European options is:

$$c(t, M_t) = M_t \exp(-r\tau) B(n-w, \tilde{\theta}) - S \exp(-i\tau) B(n-w, \theta)$$
$$p(t, M_t) = S \exp(-i\tau) \tilde{B}(n-w, \theta) - M_t \exp(-r\tau) \tilde{B}(w, n, \tilde{\theta})$$

(36)

where c(t, $M_t$) is a European call option, p(t, $M_t$) is a European put option, and $\tilde{B}(w,n,\pi) = 1 - B(w,n,\pi)$. If n → ∞, then the complementary binomial model converges to the Merton (1974) model.

## *Concept of Greek letters for the complementary binomial model*

Delta is the proportional change in the option's valuation relative to the proportional change in the underlying price:

$$\Delta = \begin{cases} \exp(-r\tau) B(n-w, \tilde{\theta}) & \text{si es una opción de compra} \\ -\exp(-r\tau) \tilde{B}(w, n, \tilde{\theta}) & \text{si es una opción de venta} \end{cases}$$

(37)

Gamma is the proportional change in delta relative to the proportional change in the underlying price:

$$\Gamma = \frac{\exp(-r\tau) B'(n-w, \tilde{\theta})}{2M_t \log(a)}$$

(38)

Nu is the proportional change in the option's valuation relative to the proportional change in the underlying volatility:

$$\nu = M_t \exp(-r\tau) \sqrt{\tau} B'(n-w, \tilde{\theta})$$

(39)

Rho is the proportional change in the option's valuation relative to the proportional change in the risk-free interest rate:

$$\rho = \begin{cases} ST \exp(-i\tau) B(n-w, \theta) & \text{si es una opción de compra} \\ -ST \exp(-i\tau) \tilde{B}(n-w, \theta) & \text{si es una opción de venta} \end{cases}$$

(40)

Theta is the proportional change in the option's valuation relative to the proportional change in time elapsed:

$$\Theta = \begin{cases} -iS\exp(-i\tau)B(n-w,\theta) - \dfrac{\sigma M_t \exp(-r\tau)B'(n-w,\tilde{\theta})}{4\sqrt{\tau}} \\ -iS\exp(-i\tau)\tilde{B}(n-w,\theta) + \dfrac{\sigma M_t \exp(-r\tau)B'(n-w,\tilde{\theta})}{4\sqrt{\tau}} \end{cases}$$

(41)

Kappa is the proportional change in the option's valuation relative to the proportional change in the settlement price:

$$\kappa = \frac{\Delta v(t, M_t)}{\Delta S} = \begin{cases} -\exp(-i\tau)B(n-w,\theta) & \text{si es una opción de compra} \\ \exp(-i\tau)\tilde{B}(n-w,\theta) & \text{si es una opción de venta} \end{cases}$$

(42)

If the binomial model satisfies the following assumption:

$$\begin{aligned} \Delta A_t &= \mu_A A_t \Delta t + \sigma_A A_t \Delta W_{A_t} \\ \Delta C_t &= \mu_C C_t \Delta t + \sigma_C C_t \Delta W_{C_t} \end{aligned}$$

(43)

where $\Delta W_t = \Delta W_{At} = \Delta W_{Ct}$ and there is a strong positive correlation in equation (43), i.e., $\rho(V_t, A_t) \to 1$, then:

$$\begin{aligned} \mu_C C_t &= \Theta + \mu_A A_t \Delta + \frac{\sigma_A^2 A_t^2 \Gamma}{2} = iS\kappa + \mu_A A_t \Delta \\ \sigma_C C_t &= \sigma_A A_t \Delta \end{aligned}$$

(44)

Therefore, it is possible to estimate $\mu_A$, and the complementary binomial model analyzed satisfies equation (10).

## *Capital as a European long position call option*

If $A_t = P_t + C_t$, then $C_t = A_t - P_t$, therefore:

$$C_t = A_t B(n-w,\tilde{\theta}) - P_t B(n-w,\theta)$$

(45)

where B(n – w,θ) = B(w,n,π) is the distribution of the complementary binomial model.

## *Liability as a European short position put option*

If $P_t = A_t - C_t$, then the structured note with European options is $P_t = A_t - c(t, A_t)$:

$$P_t = A_t - \left(A_t B\left(n-w,\tilde{\theta}\right) - P_t B\left(n-w,\theta\right)\right) = A_t \tilde{B}\left(n-w,\tilde{\theta}\right) + P_t B\left(n-w,\theta\right) \quad (46)$$

Correspondingly, the structured note with European options is $\Pi_t = P_t - p(t, A_t)$:

$$\Pi_t = P_t - \left(P_t \tilde{B}\left(n-w,\theta\right) - A_t \tilde{B}\left(n-w,\tilde{\theta}\right)\right) = P_t B\left(n-w,\theta\right) + A_t \tilde{B}\left(n-w,\tilde{\theta}\right) \quad (47)$$

## *Probability of default*

The probability of default is equivalent to the European call option being out of the money, i.e., $A_T < S$, or the European put option being in the money, i.e., $A_T < S$:

$$P\ \left(A_T < S\right) = \begin{cases} 1 - B\left(n-w,\theta\right) = 1 - \kappa \exp\left(iT\right) & \text{si es una opción de compra} \\ \tilde{B}\left(n-w,\theta\right) = \kappa \exp\left(iT\right) & \text{si es una opción de venta} \end{cases} \quad (48)$$